A study on spacelike rectifying curves in Minkowski 3-space

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Abstract

In this paper, we investigate spacelike rectifying curves via dilation of unit speed spacelike or timelike curves on Lorentzian unit spheres in Minkowski 3-space E_1^3. Then, we define a the centrode $D_\alpha(s)$ of a unit speed spacelike curve $\alpha(s)$ in E_1^3. In last section, we prove that if a unit speed spacelike curve $\alpha(s)$ in E_1^3 is neither a planar spacelike curve nor a helix, then its dilated centrode $\beta(s) = \rho_\alpha(s)D_\alpha(s)$, with dilation factor $\rho_\alpha(s)$, is always a rectifying curve, where ρ_α is the radius of curvature of α.

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1 Introduction

Let $\alpha : I \to E_1^3$ be a unit speed spacelike curve in Minkowski 3-space E_1^3 with Frenet-Serret apparatus $\{\kappa_\alpha, \tau_\alpha, T_\alpha, N_\alpha, B_\alpha\}$ where $\kappa_\alpha, \tau_\alpha, T_\alpha, N_\alpha$ and B_α denote the curvature, the torsion, the unit tangent T_α, the unit principal normal N_α and the unit binormal B_α of α, respectively.

Some important types of curves are helices (characterized by $\tau_\alpha = c\kappa_\alpha$ with a nonzero constant c), spherical curves (characterized by $(\rho_\alpha^{'}\sigma_\alpha)^{'} + \dfrac{\rho_\alpha}{\sigma_\alpha} = 0$ with $\rho_\alpha = \kappa_\alpha^{-1}$=radius of curvature, $\sigma_\alpha = \tau_\alpha^{-1}$=radius of torsion) and finally, rectifying curves given by $\dfrac{\tau_\alpha}{\kappa_\alpha} = as + b$ with constants $a \neq 0, b$.

The notion of rectifying curves was introduced by B.Y.Chen in [2]. By definition, a regular unit speed space curve $\alpha(s)$ in E^3 is called a rectifying curve, if its position vector always lies in its rectifying plane.

In [7], some characterizations of rectifying curves given by Ilarslan and Nesovic in Euclidean space. Also, Ilarslan, Nesovic and Petrovic-Torgasev have investigated rectifying curves in Minkowski space [9].

As spacelike rectifying curves are important, so is the relation between the Frenet-Serret apparatus $\{\kappa_\alpha, \tau_\alpha, T_\alpha, N_\alpha, B_\alpha\}$ of the spacelike rectifying curve $\alpha(t) = f(t)y(t)$ and that of the unit speed non-null curve $y(t)$. In this paper, we derive the Frenet-Serret apparatus of the spacelike rectifying curve $\alpha(t)$ in terms of that of the unit speed non-null curve $y(t)$.

Moreover, it is known that centrodes (i.e angular velocity vectors) play some important roles in mechanics and joint kinematics [1, 6, 15, 17, 18]. Regarding the centrode $D_\alpha = \tau_\alpha T_\alpha - \kappa_\alpha B_\alpha$ of a unit speed spacelike curve in E_1^3, it was shown in [8] that the centrode of a unit speed spacelike curve $\alpha : I \to E_1^3$ with non-zero constant curvature κ_α and non-constant torsion τ_α is a spacelike rectifying curve and vice versa.

In [4], rectifying curves as centrodes and extremal curves in Euclidean space are studied by Chen and Dillen. After them Ilarslan and Nesovic studied rectifying curves as centrodes and extremal

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curves in Minkowski 3-space [8]. In [19], extended rectifying curves in Minkowski 3-space are studied by Yılmaz, Gök and Yaylı.

In this paper, we study the spacelike rectifying curves in Minkowski 3-space. By using similar methods as in [5] we study spacelike rectifying curve as centrode and dilated centrode.

2 Preliminaries

The Minkowski 3-space E_1^3 is Euclidean 3-space E^3 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. g is defined that a vector $v \in E_1^3$ can be spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$ and null if $g(v,v) = 0$ and $v \neq 0$. Moreover, the norm(length) of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$, two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in E_1^3, can locally be spacelike, timelike or null, if all its velocity vectors $\alpha^{'}(s)$ are respectively spacelike, timelike or null. If $g(\alpha^{'}(s), \alpha^{'}(s)) = \pm 1$, the non-null curve α is said to be of unit speed (or parameterized by arc-length function s).

The Frenet frame $\{T_\alpha, N_\alpha, B_\alpha\}$ of a unit speed spacelike curve $\alpha(s)$ in E_1^3, with $g(\alpha^{''}(s), \alpha^{''}(s)) \neq 0$ for each s, is given by $T_\alpha(s) = \alpha^{'}(s), N_\alpha(s) = \dfrac{\alpha^{''}(s)}{\|\alpha^{''}(s)\|}, B_\alpha(s) = T_\alpha(s) \times N_\alpha(s)$. Let us put $g(T_\alpha, T_\alpha) = 1$ and $g(N_\alpha, N_\alpha) = \varepsilon = \pm 1$. Then $g(B_\alpha, B_\alpha) = -\varepsilon$ and the following Frenet formulas hold [12]:

$$\begin{aligned} T_\alpha^{'}(s) &= \kappa_\alpha(s) N_\alpha(s), \\ N_\alpha^{'}(s) &= -\varepsilon \kappa_\alpha(s) T_\alpha(s) + \tau_\alpha(s) B_\alpha(s), \\ B_\alpha^{'}(s) &= \tau_\alpha(s) N_\alpha(s). \end{aligned} \tag{2.1}$$

Accordingly, the Frenet frame of α satisfies the equations,

$$\begin{aligned} T_\alpha \times N_\alpha &= B_\alpha, \\ N_\alpha \times B_\alpha &= -\varepsilon T_\alpha, \\ B_\alpha \times T_\alpha &= -N_\alpha. \end{aligned} \tag{2.2}$$

We also recall that the pseudosphere of radius 1 and center at the origin is the hyperquadric in E_1^3 defined by

$$S_1^2 = \{v \in E_1^3 : g(v,v) = 1\},$$

and the pseudohyperbolic space of radius 1 and center at the origin is the hyperquadric in E_1^3 defined by

$$H_0^2 = \{v \in E_1^3 : g(v,v) = -1\}.$$

Let $\alpha = I \to E_1^3$ be a unit speed spacelike curve with curvature $\kappa_\alpha \neq 0$ and let $\{\kappa_\alpha, \tau_\alpha, T_\alpha, N_\alpha, B_\alpha\}$ be the Frenet-Serret apparatus of α. The distance function $f(s) = \|\alpha(s)\|$ of the spacelike rectifying curve satisfies

$$f(s) = \sqrt{s^2 + c_1 s + c_2},$$

where c_1 and c_2 are constants and the converse is also true. Moreover, it is also known that the unit speed spacelike curve α is a spacelike rectifying curve if and only if the ratio of torsion τ_α and curvature κ_α satisfies,

$$\frac{\tau_\alpha}{\kappa_\alpha} = as + b,$$

for some constants $a \neq 0$ and b [7].

The centrode of $\alpha : I \to E_1^3$ is defined by,

$$D_\alpha = \tau_\alpha T_\alpha - \kappa_\alpha B_\alpha,$$

which is the angular velocity vector of the motion of a mass particle along the spacelike curve α and it obeys the laws of motion:

$$\begin{aligned} T_\alpha^{'} &= D_\alpha \times T_\alpha, \\ N_\alpha^{'} &= D_\alpha \times N_\alpha, \\ B_\alpha^{'} &= D_\alpha \times B_\alpha. \end{aligned} \tag{2.3}$$

We shall find the curvature κ_y of the unit speed spacelike curve $y(t)$, which will be used subsequent work in this paper. Note that $T_y = y^{'}$ and that $\{y, y^{'}, y \times y^{'}\}$ is an orthonormal frame of E_1^3 and thus using Frenet-Serret formulae for $y(t)$ and

$$y^{''} = y + hy \times y^{'}, \tag{2.4}$$

with $h = g(y^{''}, y \times y^{'})$.

From (2.4) we have

$$T_y = y^{'}, \quad N_y = (\frac{1}{\kappa_y} y + \frac{h}{\kappa_y} y \times y^{'}). \tag{2.5}$$

It follows from the second equation in (2.5) that

$$\kappa_y = \sqrt{|h^2 - 1|}. \tag{2.6}$$

3 Some important results

In this section we recall some theorems from [8, 9], which are important for the proofs of theorems which follow.

Theorem 3.1. Let $\alpha = \alpha(s)$ be a unit speed non-null curve in E_1^3. Then the following statements hold:

(i) α is a rectifying curve with a spacelike rectifying plane if and only if, up to the parameterization, α is given by

$$\alpha(t) = \frac{a}{\cos(t)} y(t), a \in R_0^+,$$

where $y(t)$ is a unit speed spacelike curve lying in S_1^2.
(ii) α is a spacelike (resp. timelike) rectifying curve with a timelike rectifying plane and a spacelike (resp. timelike) position vector, if and only if up to the parameterization, α is given by

$$\alpha(t) = \frac{a}{\sinh(t)} y(t), a \in R_0^+,$$

where $y(t)$ is a unit speed timelike (resp. spacelike) curve lying in $S_1^2 (resp. H_0^2)$.
(iii) α is a spacelike (resp. timelike) rectifying curve with a timelike rectifying plane and a timelike (resp. spacelike) position vector, if and only if up to the parameterization, α is given by

$$\alpha(t) = \frac{a}{\cosh(t)} y(t), a \in R_0^+,$$

where $y(t)$ is a unit speed spacelike (resp. timelike) curve lying in $H_0^2 (resp. S_1^2)$ [9]

Theorem 3.2. The centrode of a unit speed spacelike curve $\alpha(s)$ in E_1^3, with constant curvature $\kappa_\alpha \neq 0$, non-constant torsion and $g(\alpha''(s), \alpha''(s)) \neq 0$ is a spacelike rectifying curve. Conversely, every unit speed spacelike rectifying curve in E_1^3, is the centrode of some unit speed spacelike curve with constant curvature $\kappa_\alpha \neq 0$ and non-constant torsion [8].

4 Spacelike rectifying curves via dilation of spacelike or timelike curves on Lorentzian unit spheres

In this section, firstly, we assume that α is a spacelike rectifying curve with a timelike position vector and $\alpha(t) = \frac{a}{\cosh(t+t_0)} y(t), a > 0, t_0 \in \mathbb{R}$ where $y(t)$ unit speed spacelike curve lying in $H_0^2 \subset E_1^3$ centered at the origin. However, if we consider an arc of the great circle, $y(t) = (\cosh t, 0, \sinh t)$ and the spacelike curve,

$$\begin{aligned} \alpha(t) &= \frac{a}{\cosh(t+t_0)} y(t), \\ &= a \left(\frac{1}{\cosh(t+t_0)} \cosh(t), 0, \frac{1}{\cosh(t+t_0)} \sinh(t) \right), \end{aligned} \tag{4.1}$$

then we get the speed υ_α and the tangent vector field T_α of α as

$$\upsilon_\alpha = \frac{a}{\cosh^2(t+t_0)}, \quad T_\alpha = (-\sinh(t_0), 0, \cosh(t_0)), \tag{4.2}$$

and therefore, the curvature κ_α of α is zero. Consequently, α cannot be a spacelike rectifying curve, as the definition of rectifying curve requires that its curvature non-zero. Therefore, not all spacelike curves that are dilations of unit speed spacelike curve $y(t)$ on H_0^2 of the type $\alpha(t) = \frac{a}{\cosh(t+t_0)} y(t)$ are rectifying curves. Therefore, the following theorem can be given according to the above findings.

Theorem 4.1. Let $y(t)$ be a unit speed spacelike curve on H_0^2 centered at the origin $0 \in E_1^3$ and let $\alpha(t) = \frac{a}{\cosh(t+t_0)} y(t)$ be a dilation of α spacelike rectifying curve with a timelike position vector and a timelike rectifying plane. The Frenet-Serret apparatus of α:

$$T_\alpha = -\sinh(t+t_0) y + \cosh(t+t_0) y',$$

$$N_\alpha = y \times y',$$
$$B_\alpha = -\cosh(t+t_0)y + \sinh(t+t_0)y',$$
$$\kappa_\alpha = \frac{1}{a}\cosh^3(t+t_0)\sqrt{\kappa_y^2+1},$$
$$\tau_\alpha = \frac{1}{a}\cosh^2(t+t_0)\sinh(t+t_0)\sqrt{\kappa_y^2+1}$$

where κ_y is the curvature of the unit speed spacelike curve $y(t)$.

Proof. By according to hypothesis of the theorem 4.1, the speed of α is given by, $v_\alpha = \frac{a}{\cosh^2(t+t_0)}$).
Since $\{y, y', y \times y'\}$ is an orthonormal frame of E_1^3 along $y(s)$, we get,

$$T_\alpha = -\sinh(t+t_0)y + \cosh(t+t_0)y'.$$

Let s be arc-length parameter for α; then we have,

$$\frac{ds}{dt} = \frac{a}{\cosh^2(t+t_0)}.$$

By differentiating of equation T_α and using equation (2.4) and Frenet-Serret formulae, we get

$$\kappa_\alpha\left(\frac{a}{\cosh^2(t+t_0)}\right)N_\alpha = \cosh(t+t_0)hy \times y',$$

with $h = g(y'', y \times y')$. Therefore, by according to equation (2.6), we get,

$$\kappa_\alpha = \frac{1}{a}\cosh^3(t+t_0)\sqrt{{\kappa_y}^2+1},$$

and

$$N_\alpha = y \times y'.$$

Now, using $B_\alpha = T_\alpha \times N_\alpha$ we get,

$$B_\alpha = -\cosh(t+t_0)y + \sinh(t+t_0)y',$$

with $y \times (y \times y') = -y'$ and $y' \times (y \times y') = -y$.
After differentiating the equation above and using equation (2.4) and Frenet-Serret formulae we get,

$$\tau_\alpha N_\alpha\left(\frac{a}{\cosh^2(t+t_0)}\right) = \sinh(t+t_0)hy \times y',$$

and it leads to

$$\tau_\alpha = \frac{1}{a}\cosh^2(t+t_0)\sinh(t+t_0)\sqrt{{\kappa_y}^2+1}.$$

Q.E.D.

Theorem 4.2. Let $y(t)$ be a unit speed timelike curve on S_1^2 centered at the origin $0 \in E_1^3$ and let $\alpha(t) = \dfrac{a}{\sinh(t+t_0)} y(t)$ be a dilation of α with a spacelike position vector and a timelike rectifying plane. The Frenet-Serret apparatus of α:

$$T_\alpha = -\cosh(t+t_0)y + \sinh(t+t_0)y^{'},$$

$$N_\alpha = y \times y^{'},$$

$$B_\alpha = -\sinh(t+t_0)y + \cosh(t+t_0)y^{'},$$

$$\kappa_\alpha = \frac{1}{a}\sinh^3(t+t_0)\sqrt{\kappa_y^2+1},$$

$$\tau_\alpha = \frac{1}{a}\sinh^2(t+t_0)\cosh(t+t_0)\sqrt{\kappa_y^2+1}$$

where κ_y is the curvature of the unit speed timelike curve $y(t)$.

Proof. The proof is made in a similar way to theorem 4.1. Q.E.D.

Corollary 4.3. Let $y(t)$ be a unit speed spacelike (resp. timelike) curve on H_0^2(resp. S_1^2) that is not an arc of the great circle, then $\alpha(t) = \dfrac{a}{\cosh(t+t_0)} y(t)$ (resp. $\alpha(t) = \dfrac{a}{\sinh(t+t_0)} y(t)$) is a rectifying spacelike curve.

5 Centrodes of unit speed spacelike curves

Let $\alpha : I \to E_1^3$ be a unit speed spacelike curve with Frenet-Serret apparatus $\{\kappa_\alpha, \tau_\alpha, T_\alpha, N_\alpha, B_\alpha\}$ and let $D_\alpha : I \to E_1^3$ be the centrode of α defined by

$$D_\alpha = \tau_\alpha T_\alpha - \kappa_\alpha B_\alpha. \tag{5.1}$$

By differentiating the expression of D_α and using Frenet-Serret formulae we get

$$D_\alpha^{'} = \tau_\alpha^{'} T_\alpha - \kappa_\alpha^{'} B_\alpha.$$

Therefore, the speed υ_D of centrode D_α is given by

$$\upsilon_D = \sqrt{|(\tau_\alpha^{'})^2 - \varepsilon(\kappa_\alpha^{'})^2|}. \tag{5.2}$$

Hence the unit tangent vector field T_D of the centrode is given by

$$T_D = \frac{\tau_\alpha^{'}}{\upsilon_D} T_\alpha - \frac{\kappa_\alpha^{'}}{\upsilon_D} B_\alpha. \tag{5.3}$$

Let s be the arc-length parameter and let κ_D denote the curvature of the centrode. Then, by differentiating equation (5.3), we get,

$$\upsilon_D \kappa_D N_D = \left(\frac{\tau_\alpha^{'}}{\upsilon_D}\right)^{'} T_\alpha + \left(\frac{\tau_\alpha^{'}\kappa_\alpha - \kappa_\alpha^{'}\tau_\alpha}{\upsilon_D}\right) N_\alpha - \left(\frac{\kappa_\alpha^{'}}{\upsilon_D}\right)^{'} B_\alpha. \tag{5.4}$$

Proposition 5.1. Let $\alpha : I \to E_1^3$ be a unit speed spacelike curve whose curvature κ_α and torsion τ_α satisfy $\kappa_\alpha \neq 0$ and $|(\tau_\alpha^{'})^2 - \varepsilon(\kappa_\alpha^{'})^2| \neq 0$. Then α is a helix if and only if its centrode is a line segment.

Proof. If the centrode α is a line segment, then equation (5.4) gives $\tau_\alpha^{'}\kappa_\alpha = \kappa_\alpha^{'}\tau_\alpha$ and this shows that $\dfrac{\tau_\alpha}{\kappa_\alpha}$ is a constant. Hence α is a helix.
Conversely, if α is a helix, then we have $\tau_\alpha = c\kappa_\alpha$ for some constant $c \neq 0$, and thus equation (5.4) gives $\kappa_D = 0$. Hence α is a line segment. Q.E.D.

Theorem 5.2. We have:
(a) Let $\alpha = \alpha(t)$ be a unit speed spacelike curve in E_1^3 whose curvature κ_α and torsion τ_α satisfy $\kappa_\alpha, \tau_\alpha \neq 0$ and $|(\tau_\alpha^{'})^2 - \varepsilon(\kappa_\alpha^{'})^2| \neq 0$. Suppose that α is not a helix. Then the centrode $D_\alpha = \tau_\alpha T_\alpha - \kappa_\alpha B_\alpha$ of α is a rectifying spacelike curve if and only if κ_α and τ_α satisfy a non-homogeneous linear equation $a\kappa_\alpha - b\tau_\alpha = c$, so that a, b, c are constants with $a^2 + b^2 \neq 0$ and $c \neq 0$.
(b) If $\kappa_\alpha^{'} \neq 0$ for $\kappa_\alpha^{'} \neq \mp\tau_\alpha^{'}$ and if the centrode $D_\alpha(t)$ of $\alpha(t)$ is a spacelike rectifying curve, then the Frenet-Serret apparatus $\{\kappa_D, \tau_D, T_D, N_D, B_D\}$ of the centrode satisfies,

$$T_D = \frac{\widehat{c}}{\sqrt{|\widehat{c}^2 - \varepsilon|}} T_\alpha - \frac{1}{\sqrt{|\widehat{c}^2 - \varepsilon|}} B_\alpha,$$
$$N_D = N_\alpha,$$
$$B_D = \frac{1}{\sqrt{|\widehat{c}^2 - \varepsilon|}} T_\alpha + \frac{\widehat{c}}{\sqrt{|\widehat{c}^2 - \varepsilon|}} B_\alpha,$$
$$\kappa_D = \frac{(\widehat{c}\kappa_\alpha - \tau_\alpha)}{\kappa_\alpha^{'}(|\widehat{c}^2 - \varepsilon|)},$$
$$\tau_D = \frac{-\widehat{c}\tau_\alpha + \kappa_\alpha}{\kappa_\alpha^{'}(|\widehat{c}^2 - \varepsilon|)},$$

where $\widehat{c} = \dfrac{a}{b}$ for $\widehat{c} \neq \mp 1$ and a, b are defined as in statement (a).
(c) If $\tau_\alpha^{'} \neq 0$ for $\kappa_\alpha^{'} \neq \mp\tau_\alpha^{'}$ and if the centrode $D(t)$ is a rectifying spacelike curve, then the Frenet-Serret apparatus $\{\kappa_D, \tau_D, T_D, N_D, B_D\}$ of the centrode satisfies,

$$T_D = \frac{1}{\sqrt{|1 + \overline{c}^2\varepsilon|}} T_\alpha + \frac{\overline{c}}{\sqrt{|1 - \overline{c}^2\varepsilon|}} B_\alpha,$$
$$N_D = N_\alpha,$$
$$B_D = \frac{\overline{c}}{\sqrt{|1 - \overline{c}^2\varepsilon|}} T_\alpha + \frac{1}{\sqrt{|1 - \overline{c}^2\varepsilon|}} B_\alpha,$$
$$\kappa_D = \frac{-\overline{c}\tau_\alpha + \kappa_\alpha}{\tau_\alpha^{'}|1 - \overline{c}^2\varepsilon|},$$
$$\tau_D = \frac{\overline{c}\kappa_\alpha - \tau_\alpha}{\tau_\alpha^{'}|1 - \overline{c}^2\varepsilon|},$$

where $\overline{c} = \dfrac{b}{a}$ for $\overline{c} \neq \mp 1$ and a, b are defined as in statement (a).

Proof. Let $\alpha(t)$ be a unit speed spacelike curve in E_1^3 whose curvature κ_α and torsion τ_α satisfy $\kappa_\alpha, \tau_\alpha \neq 0$ and $|(\tau_\alpha^{'})^2 - \varepsilon(\kappa_\alpha^{'})^2| \neq 0$. Suppose that α is not a helix. By according to equations (5.2) and (5.4), we get,

$$\upsilon_D \kappa_D \langle N_D, D_\alpha \rangle = \tau_\alpha (\frac{\tau_\alpha^{'}}{\upsilon_D})' - \varepsilon \kappa_\alpha (\frac{\kappa_\alpha^{'}}{\upsilon_D})'. \tag{5.5}$$

Since α is not a helix, $\frac{\tau_\alpha}{\kappa_\alpha}$ is non-constant. Hence, by according to proposition 5.1, we have $\kappa_D \neq 0$ and as $\sqrt{|(\tau_\alpha^{'})^2 - \varepsilon(\kappa_\alpha^{'})^2|} \neq 0$. Therefore equation (5.5) implies that,

$$\tau_\alpha (\frac{\tau_\alpha^{'}}{\sqrt{|(\tau_\alpha^{'})^2 - \varepsilon(\kappa_\alpha^{'})^2|}})' - \varepsilon \kappa_\alpha (\frac{\kappa_\alpha^{'}}{\sqrt{|(\tau_\alpha^{'})^2 - \varepsilon(\kappa_\alpha^{'})^2|}})' = 0, \tag{5.6}$$

holds identically if and only if $\langle N_D, D_\alpha \rangle = 0$ holds identically. Consequently, the centrode D_α of α is spacelike rectifying curve if and only if equation (5.6) holds.
Now we shall solve differential equation (5.6). Since $|(\tau_\alpha^{'})^2 - \varepsilon(\kappa_\alpha^{'})^2| \neq 0$, we have $(\tau_\alpha^{'})^2 - \varepsilon(\kappa_\alpha^{'})^2 \neq 0$. Consequently, we have Case 1,2,3 and 4.
Case (1): Let us assume that $\varepsilon = -1$ and $\kappa_\alpha^{'} \neq 0$. In this case we define a function g_1 by

$$g_1(t) = \tan^{-1} \left(\frac{\tau_\alpha^{'}}{\kappa_\alpha^{'}} \right). \tag{5.7}$$

From (5.7), we get

$$\sin g_1(t) = \frac{\tau_\alpha^{'}}{\sqrt{(\tau_\alpha^{'})^2 + (\kappa_\alpha^{'})^2}}, \cos g_1(t) = \frac{\kappa_\alpha^{'}}{\sqrt{(\tau_\alpha^{'})^2 + (\kappa_\alpha^{'})^2}}. \tag{5.8}$$

Hence, the equation (5.6) is equivalent to

$$(\tau_\alpha (\cos g_1(t)) - \kappa_\alpha (\sin g_1(t))) g_1^{'}(t) = 0. \tag{5.9}$$

If $\tau_\alpha (\cos g_1(t)) - \kappa_\alpha (\sin g_1(t)) = 0$ we have,

$$\frac{\tau_\alpha}{\kappa_\alpha} = \tan g_1(t) = \frac{\tau_\alpha^{'}}{\kappa_\alpha^{'}},$$

which implies that $\frac{\tau_\alpha}{\kappa_\alpha}$ is a constant. However, this is impossible since the spacelike curve α is not a helix. Therefore, we obtain $g_1^{'}(t) = 0$ from (5.9), and thus $g_1(t)$ is a constant. Consequently, $\tau_\alpha^{'} = c_1 \kappa_\alpha^{'}$ for some constant c_1. If we put $c_1 = \frac{a}{b}$ for constants a, b. Then, we obtain $a\kappa_\alpha - b\tau_\alpha = c$ for some constant c. Since $\frac{\tau_\alpha}{\kappa_\alpha}$ is non-constant, we must have $c \neq 0$ and hence $a^2 + b^2 \neq 0$.
Case (2): Let us assume that $\varepsilon = -1$ and $\tau_\alpha^{'} \neq 0$. In this case we define a function g_2 by

$$g_2(t) = \tan^{-1} \left(\frac{\kappa_\alpha^{'}}{\tau_\alpha^{'}} \right). \tag{5.10}$$

From (5.10), we get

$$\sin g_2(t) = \frac{\kappa_\alpha^{'}}{\sqrt{(\tau_\alpha^{'})^2 + (\kappa_\alpha^{'})^2}}, \quad \cos g_2(t) = \frac{\tau_\alpha^{'}}{\sqrt{(\tau_\alpha^{'})^2 + (\kappa_\alpha^{'})^2}}. \tag{5.11}$$

Similarly, we know that equation (5.6) is equivalent to,

$$(-\tau_\alpha(\sin g_2(t)) + \kappa_\alpha(\cos g_2(t)))g_2^{'}(t) = 0.$$

If $-\tau_\alpha(\sin g_2(t)) + \kappa_\alpha(\cos g_2(t) = 0$ we have

$$\frac{\kappa_\alpha}{\tau_\alpha} = \tan g_2(t) = \frac{\kappa_\alpha^{'}}{\tau_\alpha^{'}}.$$

Now, by appyling a similar argument as Case 1, we obtain $a\kappa_\alpha - b\tau_\alpha = c$ for some constants a, b, c with $a^2 + b^2 \neq 0$ and $c \neq 0$.
Case (3): Let us assume that $\varepsilon = 1$ and $\tau_\alpha^{'} \neq \pm\kappa_\alpha^{'} \neq 0$. In this case we define a function g_3 by

$$g_3(t) = \tanh^{-1}\left(\frac{\tau_\alpha^{'}}{\kappa_\alpha^{'}}\right) \tag{5.12}$$

From (5.12), we get

$$\sinh g_3(t) = \frac{\tau_\alpha^{'}}{\sqrt{|(\tau_\alpha^{'})^2 - (\kappa_\alpha^{'})^2|}}, \quad \cosh g_3(t) = \frac{\kappa_\alpha^{'}}{\sqrt{|(\tau_\alpha^{'})^2 - (\kappa_\alpha^{'})^2|}}. \tag{5.13}$$

Similarly, we know that equation (5.6) is equivalent to,

$$(\tau_\alpha \cosh g_3(t) - \kappa_\alpha \sinh g_3(t))g_3^{'}(t) = 0.$$

If $\tau_\alpha(\cosh g_3(t)) - \kappa_\alpha(\sinh g_3(t)) = 0$ we have

$$\frac{\tau_\alpha}{\kappa_\alpha} = \tanh g_3(t) = \frac{\tau_\alpha^{'}}{\kappa_\alpha^{'}}.$$

Now, by appyling a similar argument as Case 1,2 we obtain $a\kappa_\alpha - b\tau_\alpha = c$ for some constants a, b, c with $a^2 + b^2 \neq 0$ and $c \neq 0$.
Case (4): Let us assume that $\varepsilon = 1$ and $\tau_\alpha^{'} \neq \pm\kappa_\alpha^{'} \neq 0$. In this case we define a function g_4 by

$$g_4(t) = \tanh^{-1}\left(\frac{\kappa_\alpha^{'}}{\tau_\alpha^{'}}\right). \tag{5.14}$$

From (5.14), we get

$$\sinh g_4(t) = \frac{\kappa_\alpha^{'}}{\sqrt{|(\tau_\alpha^{'})^2 - (\kappa_\alpha^{'})^2|}}, \quad \cosh g_4(t) = \frac{\tau_\alpha^{'}}{\sqrt{|(\tau_\alpha^{'})^2(\kappa_\alpha^{'})^2|}}. \tag{5.15}$$

Similarly, we know that equation (5.6) is equivalent to,

$$(\tau_\alpha \sinh g_4(t) - \kappa_\alpha \cosh g_4(t))g_4'(t) = 0.$$

If $\tau_\alpha(\sinh g_4(t)) - \kappa_\alpha(\cosh g_4(t)) = 0$ we have

$$\frac{\kappa_\alpha}{\tau_\alpha} = \tanh g_4(t) = \frac{\kappa_\alpha'}{\tau_\alpha'}.$$

Now, by appyling a similar argument as Case 1,2,3, we obtain $a\kappa_\alpha - b\tau_\alpha = c$ for some constants a, b, c with $a^2 + b^2 \neq 0$ and $c \neq 0$.

Conversely, if κ_α and τ_α satisfy $a\kappa_\alpha - b\tau_\alpha = c$ for some constants a, b, c satisfying $a^2 + b^2 \neq 0$ and $c \neq 0$, then κ_α and τ_α satisfy the differential equation (5.6). This proves statement(a).

Assume that the centrode $D_\alpha(t)$ is a spacelike rectifying curve. Then statement (a) implies that the curvature and torsion of $\alpha(t)$ satisfy a non-homogeneous linear equation

$$a\kappa_\alpha - b\tau_\alpha = c,$$

for some constants a, b, c satisfying $a^2 + b^2 \neq 0$ and $c \neq 0$. In particular, we have (i) $b \neq 0$ or (ii) $a \neq 0$.

Case (i): $b \neq 0$. We find from (5.12) that

$$\tau_\alpha' = \widehat{c}\kappa_\alpha', \quad \widehat{c} = \frac{a}{b} \neq \pm 1. \tag{5.16}$$

Therefore we get from (5.2) and (5.3)

$$\upsilon_D = \kappa_\alpha' \sqrt{|\widehat{c}^2 - \varepsilon|}, \tag{5.17}$$

$$T_D = \frac{\widehat{c}}{\sqrt{|\widehat{c}^2 - \varepsilon|}} T_\alpha - \frac{1}{\sqrt{|\widehat{c}^2 - \varepsilon|}} B_\alpha.$$

Consequently, equation (5.5) reduces to

$$\kappa_\alpha' \sqrt{|\widehat{c}^2 - \varepsilon|} \kappa_D N_D = \frac{\widehat{c}\kappa_\alpha - \tau_\alpha}{\sqrt{|\widehat{c}^2 - \varepsilon|}} N_\alpha,$$

$$\kappa_D = \frac{(\widehat{c}\kappa_\alpha - \tau_\alpha)}{\kappa_\alpha'(|\widehat{c}^2 - \varepsilon|)}, \quad N_D = N_\alpha. \tag{5.18}$$

By using equations (5.15), (5.16) and $B_D = T_D \times N_D$ we get,

$$B_D = -\frac{1}{\sqrt{|\widehat{c}^2 - \varepsilon|}} T_\alpha + \frac{\widehat{c}}{\sqrt{|\widehat{c}^2 - \varepsilon|}} B_\alpha,$$

which on differentiation and using Frenet-Serret formulae gives,

$$\tau_D = \frac{-\widehat{c}\tau_\alpha + \kappa_\alpha}{\kappa_\alpha'(|\widehat{c}^2 - \varepsilon|)}.$$

This proves statement (b).

Case (ii): $a \neq 0$. From (5.14) we have $\kappa_\alpha^{'} = \bar{c}\tau_\alpha^{'}, \bar{c} = \dfrac{b}{a} \neq \pm 1$. Thus we may apply a method similar to Case (i) to obtain statement (c). Q.E.D.

Remark 5.3. The condition $a\kappa_\alpha - b\tau_\alpha = c$ with $a^2 + b^2 \neq 0$ and $c \neq 0$ given in theorem 4.1(a) has been used by Lucas and Ortega-Yages in [13] for their study of Bertrand curves in the Euclidean 3-space E^3 or Lorentz-Minkowski 3-space L^3.

Remark 5.4. Let $\alpha(t)$ be the unit speed spacelike curve of theorem 5.1 with $\kappa_\alpha^{'} \neq 0 \quad (\tau_\alpha^{'} \mp \neq \kappa_\alpha^{'})$ such that the centrode D_α of α is a spacelike rectifying curve. Then as $\tau_\alpha^{'} = \widehat{c}\kappa_\alpha^{'}$ with $\widehat{c}\mp \neq 1$, we get $\widehat{c}\kappa_\alpha = \tau_\alpha + c_1$ for a constant $c_1 \neq 0$ (as $\dfrac{\tau_\alpha}{\kappa_\alpha}$ is non-constant), that is, $\widehat{c}\kappa_\alpha - \tau_\alpha = c_1$. Moreover, the arc-length parameter s of the spacelike rectifying curve D_α satisfies $\dfrac{ds}{dt} = \kappa_\alpha^{'}\sqrt{|\widehat{c}^2 - \varepsilon|}$ which gives

$$s = \kappa_\alpha\sqrt{|\widehat{c}^2 - \varepsilon|} + b,$$

for a constant b. Thus after using the expressions of curvature and torsion of D_α we obtain,

$$\frac{\tau_D}{\kappa_D} = \frac{-\widehat{c}\tau_\alpha + \kappa_\alpha}{\widehat{c}\kappa_\alpha - \tau_\alpha} = \frac{-\widehat{c}\tau_\alpha + \kappa_\alpha}{c_1} = \frac{-\widehat{c}}{c_1}(\widehat{c}\kappa - c_1) + \frac{\kappa_\alpha}{c_1},$$

$$= (\frac{-\widehat{c}^2 + 1}{c_1})\kappa_\alpha + \widehat{c} = \frac{-\widehat{c}^2 + 1}{c_1}(\frac{s - b}{\sqrt{|\widehat{c}^2 - \varepsilon|}}) + \widehat{c},$$

$$= As + B,$$

where $A \neq 0, \quad B$ are constants. Thus, the ratio $\dfrac{\tau_D}{\kappa_D}$ is a linear function of the arc-length s as required by a spacelike rectifying curve (cf. [2, Theorem 2]).
Also, we get

$$|\tau_\alpha^2 - \varepsilon\kappa_\alpha^2| = |(\widehat{c}\kappa_\alpha - c_1)^2 - \varepsilon\kappa\alpha^2| = |\kappa_\alpha^2(\widehat{c}^2 - \varepsilon) - 2\widehat{c}c_1\kappa_\alpha + c_1^2|,$$

$$= |\widehat{c}^2 - \varepsilon|\frac{(s-b)^2}{|\widehat{c}^2 - \varepsilon|} - 2c_1\widehat{c}\frac{(s-b)}{\sqrt{|\widehat{c}^2 - \varepsilon|}} + c_1^2,$$

$$= s^2 + \lambda_1 s + \lambda_2,$$

where λ_1, λ_2 are constants in E_1^3. Therefore the distance function $f(s) = \|D_\alpha\|$ of the spacelike rectifying curve D_α satisfies $f(s) = \sqrt{s^2 + \lambda_1 s + \lambda_2}$, as required by a spacelike rectifying curve.

Note that similar arguments hold for a unit speed spacelike curve $\alpha(t)$ with $\tau_\alpha^{'} \neq 0$ (instead of $\kappa_\alpha^{'} \neq 0$) and with a spacelike rectifying centrode.

Corollary 5.5. Let $\alpha : I \to E_1^3$ be a unit speed spacelike curve whose curvature κ_α and torsion τ_α satisfy $\kappa_\alpha^{'} \neq 0$, $\frac{\tau_\alpha}{\kappa_\alpha}$ non-constant, and $\tau_\alpha^{'} = \widehat{c}\kappa_\alpha^{'}, \widehat{c} \neq \mp 1$ being a constant. Then the centrode D_α of α is a spacelike rectifying curve with curvature $\kappa_D = \pm\frac{a}{\kappa_\alpha^{'}}$ and torsion $\tau_D = \frac{b_1 s + c_1}{\kappa_\alpha^{'}}$ for some constants $a, b_1 \neq 0$ and c_1.

Proof. Under the hypothesis of the theorem, we have

$$\frac{\tau_D}{\kappa_D} = \frac{-\widehat{c}\tau_\alpha + \kappa_\alpha}{\widehat{c}\kappa_\alpha - \tau_\alpha},$$

which implies

$$1 - (\frac{\tau_D}{\kappa_D})^2 = \frac{\widehat{c}^2 - 1}{(\widehat{c}\kappa_\alpha - \tau_\alpha)^2}(\kappa_\alpha^2 - \tau_\alpha^2). \tag{5.19}$$

Now, as $\tau_\alpha^{'} = \widehat{c}\kappa_\alpha^{'}$, we have $\widehat{c}\kappa_\alpha - \tau_\alpha = c_0$, where c_0 is non-zero constant (as $\frac{\tau_\alpha}{\kappa_\alpha}$ is non-constant), and from equation (5.19), we get

$$\frac{\kappa_D^2 - \tau_D^2}{\kappa_\alpha^2 - \tau_\alpha^2} = \frac{\widehat{c}^2 - 1}{c_0^2}\kappa_D^2. \tag{5.20}$$

By according to equation (5.19), we find

$$\kappa_D^2 - \tau_D^2 = \frac{\widehat{c}^2 - 1}{(\kappa_\alpha^{'})^2(\widehat{c}^2 - \varepsilon)^2}(\kappa_\alpha^2 - \tau_\alpha^2),$$

that is,

$$\frac{\kappa_D^2 - \tau_D^2}{\kappa_\alpha^2 - \tau_\alpha^2} = \frac{\widehat{c}^2}{(\kappa_\alpha^{'})^2(\widehat{c}^2 - \varepsilon)}.$$

After combining this equation with equation (5.20), we get

(i) If T_α, N_α are spacelike vectors and B_α is a timelike vector, i.e. $\varepsilon = 1$, then we have

$$\kappa_D^2 = \frac{c_0^2}{(\kappa_\alpha^{'})^2(\widehat{c}^2 - 1)^2} = \frac{c_2^2}{(\kappa_\alpha^{'})^2},$$

where c_2 is a non-zero constant. Therefore we obtain the required expression for the curvature κ_D of the spacelike rectifying curve D_α.

(ii) If T_α, B_α are spacelike vectors and N_α is a timelike vector, i.e. $\varepsilon = -1$, then we have

$$\kappa_D^2 = \frac{c_0^2}{(\kappa_\alpha^{'})^2(\widehat{c}^2 + 1)^2} = \frac{c_3^2}{(\kappa_\alpha^{'})^2},$$

where c_3 is a non-zero constant. Therefore we obtain the required expression for the curvature κ_D of the spacelike rectifying curve D_α.

The arc-length function s of the spacelike rectifying curve is given by $s = \kappa_\alpha\sqrt{|\widehat{c}^2 - \varepsilon|} + c_3$ for a constant c_3 and consequently, using the expression for torsion τ_D in statement (b) of theorem 5.1, we derive that

$$\tau_D = \frac{-\widehat{c}\tau_\alpha + \kappa_\alpha}{\kappa_\alpha^{'}|\widehat{c}^2 - \varepsilon|} = \frac{-\widehat{c}(\widehat{c}\kappa_\alpha - c_0) + \kappa_\alpha}{\kappa_\alpha^{'}|\widehat{c}^2 - \varepsilon|},$$

$$\frac{(-\widehat{c}^2 + 1)\kappa_\alpha + \widehat{c}c_0}{\kappa_\alpha^{'}|\widehat{c}^2 - \varepsilon|} = \frac{1}{\kappa_\alpha^{'}}(-\widehat{c}^2 + 1)[\frac{(s - c_3)}{\sqrt{|\widehat{c}^2 - \varepsilon|}|\widehat{c}^2 - \varepsilon|} + \frac{\widehat{c}c_0}{\widehat{c}^2 - \varepsilon}].$$

(i) If T_α, N_α are spacelike vectors and B_α is a timelike vector, i.e. $\varepsilon = 1$, then we have

$$\tau_D = -\frac{1}{\kappa_\alpha^{'}}[\frac{(s - c_3)}{\sqrt{\widehat{c}^2 - 1}} - \widehat{c}c_0,$$

$$= \frac{c_4 s + c_5}{\kappa_\alpha^{'}},$$

where $c_4 \neq 0$ and c_5 are constants.

(ii) If T_α, B_α are spacelike vectors and N_α is a timelike vector, i.e. $\varepsilon = -1$, then we have

$$\frac{1}{\kappa_\alpha^{'}}[\frac{(-\widehat{c}^2 + 1)(s - c_3)}{\sqrt{\widehat{c}^2 + 1}(\widehat{c}^2 + 1)} - \frac{\widehat{c}c_0}{(\widehat{c}^2 + 1)},$$

$$= \frac{c_6 s + c_7}{\kappa_\alpha^{'}},$$

where $c_6 \neq 0$ and c_7 are constants. Q.E.D.

Remark 5.6. A result similar to corollary 5.1 holds for a unit speed spacelike curve satisfying $\tau_\alpha^{'} \neq 0$, $\frac{\tau_\alpha}{\kappa_\alpha}$ non-constant, and $\kappa_\alpha^{'} = \bar{c}\tau_\alpha^{'}$ with constant $\bar{c}$.

6 Dilated centrodes as spacelike rectifying curves

Finally, we study the dilated centrode of a unit speed twisted spacelike curve $\alpha : I \to E_1^3$ with $\kappa_\alpha > 0, \tau_\alpha \neq 0$. The dilated centrode of $\alpha(t)$ is defined by

$$\beta(t) = \rho_\alpha(t)D_\alpha(t) = \frac{\tau_\alpha(t)}{\kappa_\alpha(t)}T_\alpha - B_\alpha, \tag{6.1}$$

where $D_\alpha = \tau_\alpha T_\alpha - \kappa_\alpha B_\alpha$ is the centrode and $\rho_\alpha = \kappa_\alpha^{-1}$ is the radius of curvature of α spacelike curve.

Theorem 6.1. Let $\alpha : I \to E_1^3$ be a unit speed spacelike curve with curvature $\kappa_\alpha > 0$. If α is neither a planar spacelike curve nor a helix, then the dilated centrode $\beta(t) = \rho_\alpha(t)D_\alpha(t)$ of is α spacelike rectifying curve.

Proof. Let $\alpha : I \to E_1^3$ be a unit speed spacelike with curvature $\kappa_\alpha > 0$. Suppose that α is neither a planar spacelike curve nor a helix. Consider the dilated centrode of defined by (6.1). By differentiating equation (6.1) and using Frenet-Serret formulae, we get

$$(\beta)^{'} = (\rho_\alpha\tau_\alpha)^{'}T_\alpha,$$

and since α is not a helix, we have $(\rho_\alpha \tau_\alpha)' \neq 0$. Thus the dilated centrode $\beta(t)$ is a regular spacelike curve whose speed υ_β and unit tangent vector field T_β are given respectively by

$$\begin{aligned} \upsilon_\beta &= (\rho_\alpha \tau_\alpha)', \\ T_\beta &= T_\alpha. \end{aligned} \tag{6.2}$$

Let $\{\kappa_\beta, \tau_\beta, T_\beta, N_\beta, B_\beta\}$ be the Frenet-Serret apparatus of β. Then after differentiating equation (6.2) we find,

$$\kappa_\beta (\rho_\alpha \tau_\alpha)' N_\beta = \kappa_\alpha N_\alpha,$$

and

$$\kappa_\beta = \frac{\kappa_\alpha}{(\rho_\alpha \tau_\alpha)'}, \quad N_\beta = N_\alpha. \tag{6.3}$$

Using equations (6.1) and (6.3) we find that $\langle \beta, N_\beta \rangle = 0$, that is, the dilated centrode β is a spacelike rectifying curve. This proves the theorem.
Note that if follows from (6.2) and (6.3) that $B_\beta = B_\alpha$ and that,

$$\tau_\beta = \frac{\tau_\alpha}{(\rho_\alpha \tau_\alpha)'}. \tag{6.4}$$

Q.E.D.

Remark 6.2. Let $\alpha : I \to E_1^3$ be a unit speed spacelike curve with Frenet-Serret apparatus $\{\kappa_\alpha, \tau_\alpha, T_\alpha, N_\alpha, B_\alpha\}$, $\kappa_\alpha > 0, \tau_\alpha \neq 0$ and $(\rho_\alpha \tau_\alpha)' \neq 0$. Then according to theorem 6.1, the spacelike curve $\beta(t) = \rho_\alpha \tau_\alpha T_\alpha - B_\alpha$ is a spacelike rectifying curve whose Frenet-Serret apparatus $\{\kappa_\beta, \tau_\beta, T_\beta, N_\beta, B_\beta\}$ is given by equations (6.2) and (6.3). If s is the arc-length of β, then by equation (6.2) we have $s = \pm \rho_\alpha \tau_\alpha + c$ for a constant c. The distance function $f(s) = \|\beta(s)\|$ in view of equation (6.1) is given by

$$f(s) = \sqrt{|(\rho_\alpha \tau_\alpha)^2 - \varepsilon|} = \sqrt{(s-c)^2 + (-\varepsilon)} = \sqrt{s^2 + c_1 s + c_2},$$

where $c_1 = -2c, c_2 = c^2 - \varepsilon$. Hence the distance function $f(s)$ of the spacelike rectifying curve β has the form described in [2].
Similarly, using equation (6.3) and (6.4), we get,

$$\frac{\tau_\beta}{\kappa_\beta} = \frac{\tau_\alpha}{\kappa_\alpha} = \mp (s-c) = as + b,$$

where $a \neq 0, b$ are constants. Therefore the ratio of torsion and curvature of the spacelike rectifying curve β is the linear function of the arc-length [2].

Example 6.3. Let the curve $\alpha : I \to E_1^3$ defined by

$$\alpha(t) = (\cosh t, \sinh t, \frac{t^2}{2}).$$

The speed, unit tangent vector field, and curvature of α given respectively by

$$\upsilon_\alpha = \sqrt{t^2+1}, \quad T_\alpha = \frac{1}{\sqrt{t^2+1}} (\sinh(t), \cosh t, t), \quad \kappa_\alpha = \frac{\sqrt{t^2+2}}{(t^2+1)^{\frac{3}{2}}}$$

for $t > 1$ and the principal normal vector field

$$N_\alpha = \frac{1}{\sqrt{(t^2+1)(t^2+2)}}((t^2+1)\cosh t - t\sinh t, (t^2+1)\sinh t - t\cosh t, 1).$$

The binormal vector field and torsion of α are

$$B_\alpha = \frac{1}{\sqrt{t^2+2}}(t\sinh t - \cosh t, t\cosh t - \sinh t, -1) \quad , \tau_\alpha = \frac{-t}{t^2+2}.$$

Consequently, we have $\kappa_\alpha > 0$ and that α is neither a planar curve nor a helix. Hence, by theorem 6.1, the dilated centrode

$$\beta(t) = \frac{1}{\kappa_\alpha} D_\alpha(t) = \left(\frac{\tau_\alpha}{\kappa_\alpha}\right) T_\alpha - B_\alpha,$$

of α is a spacelike rectifying curve.

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